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Thermal instability and heat transfer in a multi-layer system subjected to uniform heat flux from below

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Abstract—Thermal instability and heat transfer in a system consisting of multiple layers separated by solid partitions are studied. A closed form solution is derived when the system is subjected to a uniform heat flux from below and adiabatic boundary conditions at two ends. For limiting cases of single and double layers, the critical Rayleigh numbers agree with previously published results. The solid partitions are found to be most effective in suppressing flow and heat transfer when they are equally distributed in a single fluid. The equivalent Rayleigh number is shown to be the weighted average of the product of the thermal conductivity and the height of each layer.

1. INTRODUCTION

THE STUDY of thermal instability and heat transfer in a system with one or more layers heated from below finds many important applications in metal casting operation, crystal growth, heat transfer in air pockets of heat exchangers, air in the multiple layer glass window, solar energy collector, and in many other geological, chemical and astrophysical systems. The theoretical study of the above-mentioned problem has a direct link to the origination of turbulence, non-linear instability of the flow, bifurcation and frequency doubling. In many situations, one may be concerned about the conditions which lead to the motion in the quiescent fluid, and the heat transfer mechanism switching from conduction to convection. In the other cases, one may be interested to know how to suppress the possible convection due to bottom heating. Even though single layer systems [1–5] and double layer systems [6–10] heated from below have received a great deal of attention in the past, there have been very few studies related to the thermal instability and heat transfer phenomena in a system with more than two layers.

The objective of this paper is to study the onset of a system consisting of multi-layer fluids separated by solid partitions, and the consequent heat transfer increase due to the fluid motion. The system under consideration is displayed in Fig. 1. Here a steady heat flux through the bottom is supplied to the system which contains M layers of fluid and $(M-1)$ layers of solid. The total number of layers is $N = 2M-1$. Each layer is a shallow cavity, so that its height, h_i, satisfies $h_i \ll L$. The approximation of parallel flow in each fluid layer enables the stream-function and temperature to be expressed as fourth- and fifth-order polynomials, respectively. Using appropriate boundary conditions at the interfaces, the unknown coefficients in the polynomials can be determined. With energy conservation across all the layers, the critical heat flux can be found for the onset of motion, and the consequent heat transfer rate can thus be obtained. The closed form solution is especially interesting since it will provide an easy approach to this complex problem. Attention will be drawn to physical aspects of the influence of the conductivity, thickness and relative position of the solid partitions in the system.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

A two-dimensional shallow cavity filled with multiple layers of fluids separated by solid partitions is considered. The system is heated by a uniform heat flux q from below, and the end walls are adiabatic. Let subscript i denote the ith layer. For the fluid layers, with the Boussinesq approximation and the approximation of constant physical properties, one can write the continuity equation, the Navier–Stokes equations, and the energy equation for the steady state as follows:

$$(u_i)_{,x}+(v_i)_{,y} = 0 \quad (1)$$

$$u_i(u_i)_{,x}+v_i(u_i)_{,y} = -\frac{1}{\rho_i}(p_i)_{,x}+\nu_i[(u_i)_{,xx}+(u_i)_{,yy}] \quad (2)$$

$$u_i(v_i)_{,x}+v_i(v_i)_{,y} = -\frac{1}{\rho_i}(p_i)_{,y}+\nu_i[(v_i)_{,xx}+(v_i)_{,yy}]+g\beta_i\Delta T \quad (3)$$

$$u_i(T_i)_{,x}+v_i(T_i)_{,y} = \alpha_i[(T_i)_{,xx}+(T_i)_{,yy}]. \quad (4)$$

For the solid layer, the energy equation takes the form

$$(T_i)_{,xx}+(T_i)_{,yy} = 0. \quad (5)$$

NOMENCLATURE

a, b integration constants in the energy equation
A aspect ratio for the two-layer system (equation (55b))
C temperature gradient in the x-direction
c_p specific heat
g gravitational acceleration
h height of system
h_i height of layer i
I, J, K constants in equations (37)–(39)
$J(f, g)$ Jacobian, $f_x g_y - f_y g_x$
L length of cavity
M number of fluid layers
N total number of layers
Nu Nusselt number
p static pressure
q heat flux
R ratio defined in equation (62)
Ra Rayleigh number
T temperature
T_R reference temperature
u, v velocity components in the x- and y-directions, respectively
x, y Cartesian coordinates
X_B fluid to solid conductivity ratio.

Greek symbols
α thermal diffusivity
β volumetric expansion coefficient
Θ temperature varying in y
ν kinematic viscosity
ρ density
ψ stream-function.

Subscripts
1 fluid properties
c critical quantities
i ith layer ($i = 1, 2, \ldots, N$)
t time derivative
, spatial derivative.

Other symbol
∇^2 Laplacian operator.

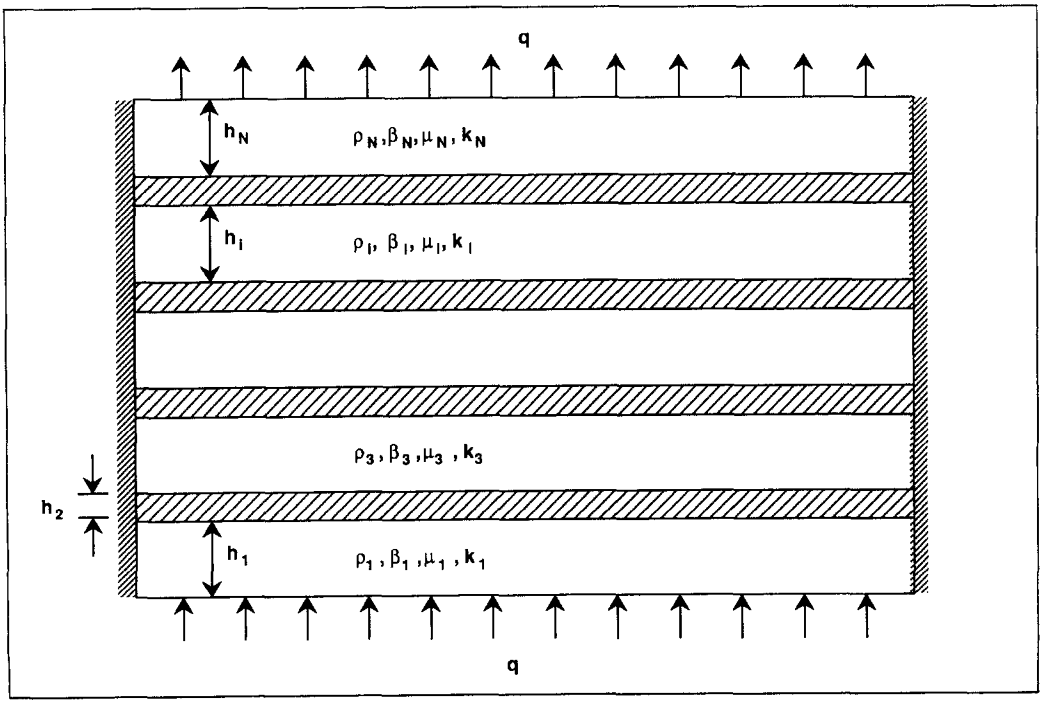

FIG. 1. Multi-layer system.

To unify our solution in a simpler fashion, the solid is taken as fluid with infinitely large viscosity, so that equations (1)–(4) will be reduced to

$$u_i = 0 \tag{6}$$

$$v_i = 0 \tag{7}$$

$$(T_i)_{,xx} + (T_i)_{,xx} = 0 \tag{8}$$

which is just what we need for the solid equation. As a result of the separation of fluids by solid partitions, the velocities at interfaces are zero. By this we can write the boundary conditions as:

at interface $(i, i-1)$

$$u_i = 0, \quad v_i = 0, \quad u_{i-1} = 0, \quad v_{i-1} = 0,$$
$$T_i = T_{i-1}, \quad k_i(T_i)_{,y} = k_{i-1}(T_{i-1})_{,y} \tag{9}$$

at interface $(i, i+1)$

$$u_i = 0, \quad v_i = 0, \quad u_{i+1} = 0, \quad v_{i+1} = 0,$$
$$T_i = T_{i+1}, \quad k_i(T_i)_{,y} = k_{i+1}(T_{i+1})_{,y} \tag{10}$$

at two end walls of $x = 0$ and L

$$u_i = 0, \quad v_i = 0, \quad k_i(T_i)_{,x} = 0. \tag{11}$$

A stream-function is introduced at this stage, so that the continuity equation is always satisfied

$$u_i = (\psi_i)_{,y}, \quad v_i = -(\psi_i)_{,x}. \tag{12}$$

By cross differentiation of equations (2) and (3), one can eliminate the pressure terms in the momentum equations, resulting in

$$J(\psi_i, \nabla^2\psi_i) = v_i\nabla^4\psi_i - g\beta_i(T_i)_{,x}. \tag{13}$$

The energy equation (4) can also be expressed as

$$J(\psi_i, T_i) = -\alpha_i\nabla^2 T_i \tag{14}$$

where

$$J(f, g) = f_x g_y - f_y g_x. \tag{15}$$

The corresponding boundary conditions for the stream-function will be:

at interface $(i, i-1)$

$$\psi_i = (\psi_i)_{,y} = 0, \quad \psi_{i-1} = (\psi_{i-1})_{,y} = 0 \tag{16}$$

at interface $(i, i+1)$

$$\psi_i = (\psi_i)_{,y} = 0, \quad \psi_{i+1} = (\psi_{i+1})_{,y} = 0 \tag{17}$$

at two end walls of $x = 0$ and L

$$\psi_i = (\psi_i)_{,x} = 0. \tag{18}$$

3. ANALYTICAL SOLUTION

The solution of equations (13) and (14) with boundary conditions (16)–(18) for an arbitrary number of layers is difficult and time consuming. Since different layers may have different physical properties, they will enforce different length and velocity scales. The present study seeks the solution of a unicellular pattern in each fluid layer as shown in Fig. 2. For this unicellular flow pattern, apart from the end walls, a parallel flow assumption can describe the basic flow features well, and will be used in this paper. It is shown that when heat flux is increased to a certain level, multiple solutions may become possible. However, we will narrow down to the unicellular flow pattern to make the simplest closed form solution possible.

The parallel flow approximation is originally from the study of infinitely extended thin layers, and has been used to study natural convection in a shallow cavity heated at two ends by Cormack *et al.* [11] and by Bejan and Tien [12] and in a shallow horizontal cylindrical cavity heated at two ends by Bejan and Tien [13]. It has been extended to a shallow inclined cavity to study the multiple steady states of flow by Vasseur *et al.* [14] and to the Bénard–Marangoni instability in a two-layer system by Yang and Yang [10]. This parallel flow is an onset mode, which has zero wave number, due to constant heat flux from below [5]. For the constant temperature condition, the wave number of the onset mode is larger than zero [15]. As shown in Section 4, the approximation is valid for the present situation. In practice, the constant heat flux boundary condition at the bottom can be supplied electrically by passing the necessary amount of current through heating coils suitably placed below the surface [1]. The energy thus supplied can be conveniently measured by the square of the current times the resistance of the circuit. If the top surface is a free surface, the energy conducted up to the surface must be carried away by convective–conductive transport to the environment. A constant heat flux boundary can be recognized when the Biot number is sufficiently high [5]. In the parallel flow approximation, the flow is assumed to depend on y only, so that $v_i = 0$, $u_i = u_i(y)$. The temperature field is assumed to be a superposition of a linear function of x and an unknown function of y. With this approximation, obviously the boundary conditions of equation (11) in the x-direction cannot be exactly satisfied, instead an integral condition on the average flux at any y section is used as in the following:

$$\sum_{i=1}^{N} \int_0^{h_i} [\rho_i c_{pi} u_i T_i - k_i(T_i)_{,x}]\, dy = 0. \tag{19}$$

The above condition for a single layer has been discussed and used by Vasseur *et al.* [14] and Bejan [16]. For the uniform flux heating in a single layer as demonstrated by Vasseur *et al.* [14], the parallel flow approximation gives a reasonable prediction for the flow and heat transfer in a cavity with an aspect ratio less than 0.5 (shallow cavity). With this approximation we have

$$\psi_i = \psi_i(y) \tag{20}$$

$$T_i = C_i x + \Theta_i(y) \tag{21}$$

where the C_i's are the unknown constant temperature gradients in the x-direction in the ith layer. As the

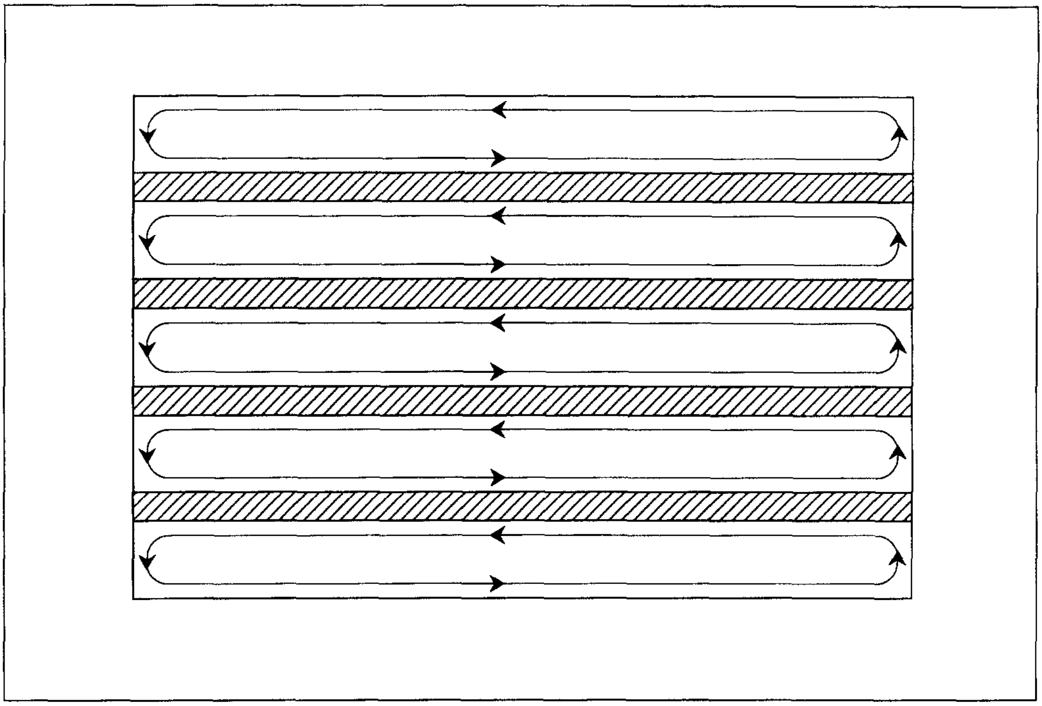

FIG. 2. Sketch of parallel flow in the system.

continuity of the temperature at the interface requires, it can be shown that at interface $(i, i-1)$ for any x

$$T_i = T_{i-1} \tag{22}$$

will result in

$$C_1 = C_2 = \cdots = C_i = \cdots = C. \tag{23}$$

With conditions (20)–(23), the governing equations can be simplified to

$$(\psi_i)_{,yyyy} = \frac{g\beta_i}{v_i} C \tag{24}$$

$$(\Theta_i)_{,yy} = \frac{C}{\alpha_i} (\psi_i)_{,y}. \tag{25}$$

If y is taken as the local coordinate with respect to the ith layer, the boundary conditions are simply

$$\text{at} \quad y = 0, \quad \psi_i = (\psi_i)_{,y} = 0, \quad \Theta_i = \Theta_{i-1}$$
$$k_i(\Theta_i)_{,y} = k_{i-1}(\Theta_{i-1})_{,y} \tag{26}$$
$$\text{at} \quad y = h_i, \quad \psi_i = (\psi_i)_{,y} = 0, \quad \Theta_i = \Theta_{i+1},$$
$$k_i(\Theta_i)_{,y} = k_{i+1}(\Theta_{i+1})_{,y}. \tag{27}$$

The solutions of equations (24) and (25) are fourth- and fifth-order polynomials, respectively. With boundary conditions (26) and (27), we have

$$\psi_i = \frac{g\beta_i C}{24 v_i} y^2 (y - h_i)^2 \tag{28}$$

$$\Theta_i = \frac{g\beta_i C^2}{24 v_i \alpha_i} \left(\frac{y^5}{5} - \frac{y^4 h_i}{2} + \frac{y^3 h_i^2}{3} \right) + a_i y + b_i \tag{29}$$

where a_i and b_i are integration constants. Continuity of heat flux at interfaces $(i, i-1)$ and $(i, i+1)$ leads to

$$k_{i-1} a_{i-1} = k_i a_i = k_{i+1} a_{i+1}. \tag{30}$$

For the first layer, where $i = 1$, a uniform heat flux at the bottom surface gives

$$k_1 a_1 = -q. \tag{31}$$

Combining with equation (30), one has

$$k_1 a_1 = k_2 a_2 = \cdots = k_i a_i = \cdots = k_N a_N = -q. \tag{32}$$

Therefore

$$a_i = -\frac{q}{k_i}, \quad i = 1, \ldots, N. \tag{33}$$

Based on the continuity of temperature at interface $(i, i-1)$, it is seen that

$$b_i = \Theta_{i-1}(y)|_{y = h_{i-1}}. \tag{34}$$

If the temperature at the bottom is selected as a reference temperature, T_R, it is immediately found that

$$b_1 = T_R \tag{35}$$

and b_i, for $i = 2, 3, \ldots, N$, can be deduced from equation (34).

At this point, the only unknown in equations (28) and (29) is constant C, which can be determined by equation (19). It should be noted that equation (19) represents the energy conservation across all N layers at any y section. If it is applied to each individual layer, the constant C will be different for each layer, which violates the requirement of temperature continuity at the interface set up by equations (22) and (23).

Now by making use of equation (19) we are able to find a condition that gives the value of C

$$KC^3-(I-J)C=0. \quad (36)$$

Here K, I and J are given by

$$K=\sum_{i=1}^{N}\int_0^{h_i}\frac{k_i}{\alpha_i^2}\left(\frac{\psi_i}{C}\right)^2 \mathrm{d}y=\sum_{i=1}^{N}Ra_i^2\left(\frac{k_i}{q}\right)^2\frac{k_ih_i}{34\,560} \quad (37)$$

$$I=\sum_{i=1}^{N}\int_0^{h_i}\frac{q}{\alpha_i}\left(\frac{\psi_i}{C}\right)\mathrm{d}y=\sum_{i=1}^{N}\frac{Ra_i\,k_ih_i}{720} \quad (38)$$

$$J=\sum_{i=1}^{N}k_ih_i \quad (39)$$

where Ra_i is the local Rayleigh number which is based on the heat flux, physical properties and the height of the fluid layer

$$Ra_i=\frac{g\beta_ih_i^3}{v_i\alpha_i}\cdot\frac{qh_i}{k_i}. \quad (40)$$

For a solid, since $\beta_i=0$ and $v_i\rightarrow\infty$, $Ra_i=0$. The solutions for C are then

$$C=\begin{cases}0\\ \pm(I-J)^{1/2}/K^{1.2}\end{cases}. \quad (41)$$

Since K is always positive, $C=0$ is the only real root when $I<J$. It implies a quiescent state. When $I>J$, two sets of convection cells bifurcate from the rest state. The onset of motion is determined by the difference between I and J. The critical heat flux can be found when

$$I=J \quad (42)$$

i.e. when

$$\sum_{i=1}^{N}k_ih_i\left(\frac{Ra_i}{720}-1\right)=0. \quad (43)$$

A critical heat flux can be thus determined

$$q_c=720\frac{\sum_{i=1}^{N}k_ih_i}{\sum_{i=1}^{N}\frac{g\beta_ih_i^5}{v_i\alpha_i}}. \quad (44)$$

Once $I>J$, we have, from equation (41)

$$C=\pm 6q\left[\frac{14\sum_{i=1}^{N}(Ra_i-720)k_ih_i}{\sum_{i=1}^{N}Ra_i^2\,k_i^3h_i}\right]^{1/2}. \quad (45)$$

With C given by equation (45), velocity and temperature profiles are obtained as

$$u_i=\frac{g\beta_iC}{12v_i\alpha_i}y(y-h_i)(2y-h_i) \quad (46)$$

$$v_i=0 \quad (47)$$

$$T_i=Cx+\frac{g\beta_iC^2}{24v_i\alpha_i}\left(\frac{y^5}{5}-\frac{y^4h_i}{2}+\frac{y^3h_i^2}{3}\right)-\frac{qy}{k_i}+T_{i-1}(y)|_{y=h_{i-1}}. \quad (48)$$

The heat transfer through the system represented by Nusselt number is then

$$Nu=\frac{\sum_{i=1}^{N}\frac{h_i}{k_i}}{\sum_{i=1}^{N}\left[\frac{h_i}{k_i}-\frac{g\beta_ih_i^5C^2}{720v_i\alpha_iq}\right]}. \quad (49)$$

4. RESULTS AND DISCUSSION

The above derived critical heat flux for the onset of motion and heat transfer rate can be applied to an arbitrary number of fluid layers separated by solid partitions subjected to a uniform heat flux from below. In this section, we will apply the solution to some special systems consisting of multiple fluid layers.

4.1. *Single fluid layer*

Dealing with the single fluid layer, the critical Rayleigh number from equation (43) is reduced to $Ra=720$, which agrees with the previous result given by Sparrow *et al.* [5]. Lienhard [17] developed a technique for predicting the stability limit of conductively coupled horizontal layers heated from below and cooled from above. He also predicted a critical Rayleigh number of 720 for a constant heat flux condition. The Nusselt number as for a Rayleigh number greater than the critical value of 720 is

$$Nu=\frac{10Ra}{3Ra+5040}. \quad (50)$$

Obviously, when the Rayleigh number is less than 720, the Nusselt number is less than 1.0 which is physically unrealistic. When Ra becomes very large, the Nusselt number approaches to a limit of 10/3. Equation (50) for Nusselt number and its limiting value of 10/3 agrees with those of Vasseur *et al.* [14] when the titled angle in their study is set to zero.

As for the situation of uniform temperatures at the top and bottom walls, the critical Rayleigh number is 1708 [1, 7] which is higher than that of the uniform heat flux case. This is due to the fact that a uniform temperature wall dissipates the thermal disturbance easier compared to that of a uniform heat flux wall. To compare behaviors of Nusselt number dependence on Rayleigh number between the different boundary conditions, the analytical solution and experimental data for a uniform temperature wall [1] and the pre-

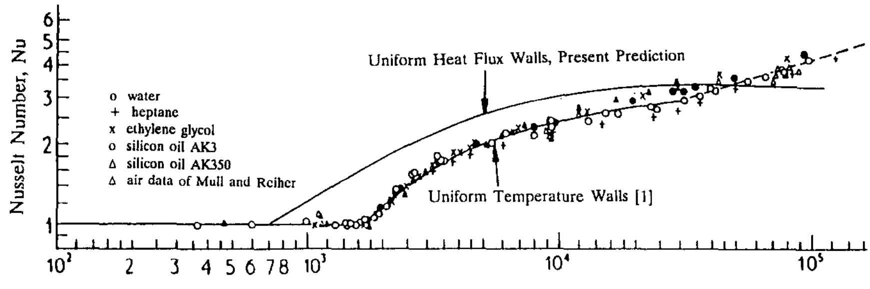

FIG. 3. Comparison of isothermal wall to uniform heat flux wall.

sent solution for constant heat flux wall are shown in Fig. 3. The general trends of Nusselt number with Rayleigh number for both boundary conditions are the same, except the values where Nu starts to be larger than unity are different. It is also seen that at high Rayleigh number (5×10^4) the slope of the Nu curve has an abrupt change for isothermal walls. This is related to the transition of the preferred mode. Similarly, one may expect that the parallel flow mode is valid only up to a certain Rayleigh number, which is subjected to further study.

4.2. *Single fluid separated by a solid partition*

Let a solid partition be inserted into the above single fluid layer so that the heights of the top and bottom layers are h_1 and h_3, respectively. The thickness of the solid and its conductivity are h_2 and k_2. With subscript '1' representing the properties of fluid, the critical heat flux determined from equation (44) becomes

$$q_c = 720\frac{k_1(h_1+h_3)+k_2h_2}{\frac{g\beta_1}{\nu_1\alpha_1}(h_1^5+h_3^5)} \qquad (51)$$

and the heat transfer rate is

$$Nu = \frac{\frac{h_1+h_3}{k_1}+\frac{h_2}{k_2}}{\frac{h_1+h_3}{k_1}+\frac{h_2}{k_2}-\frac{g\beta_1(h_1^5+h_3^5)C^2}{720\nu_1\alpha_1 q}}. \qquad (52)$$

When h_1+h_3 = constant, the critical heat flux q_c has a maximum value if $h_1 = h_3$. In other words, when the solid partition is located at the center of the fluid, the efficiency in suppressing motion is optimum. In this case the critical heat flux becomes

$$q_c = 360\frac{\nu_1\alpha_1(2k_1h_1+k_2h_2)}{g\beta_1h_1^5}. \qquad (53)$$

By rearranging equation (53), one can obtain a critical Rayleigh number based on the properties and height of the fluid layer, as

$$Ra_{c1} = 720\left(1+\frac{k_2h_2}{2k_1h_1}\right). \qquad (54)$$

Let A be the mid-layer to fluid layer aspect ratio

$$A = \frac{h_2}{2h_1} \qquad (55)$$

and X_B the fluid to solid layer conductivity ratio

$$X_B = \frac{k_1}{k_2}. \qquad (56)$$

Equation (54) can be expressed as

$$Ra_{c1} = 720\left(1+\frac{A}{X_B}\right). \qquad (57)$$

It is seen that Ra_{c1} is a function of A/X_B. Both Ulrich [18] and Lienhard [17] found that, for a small value of A, Ra_c was a function of the single parameter A/X_B. The present analysis agrees with their findings. Lienhard listed Ra_c for different A and X_B [17]. For uniform heat flux, the comparison of his data with the present prediction is listed in Table 1. The comparison, as shown in Table 1, is favorable for small values of A and large values of X_B, or for small values of A/X_B.

When $A/X_B < 0.333$, the error is less than 0.3%. This suggests that for a two-layer system with a solid partition, the preferred mode on onset motion is parallel flow if $A/X_B < 1/3$. Table 1 also proves the validity of parallel flow to the present uniform heat flux system.

Table 1. Comparison of critical Rayleigh number for two-layer system with solid partition. Present prediction (PP); Lienhard's prediction (LP) [17]

X_B/A		0.001	0.01	0.1
0.03	PP	744.00	960.00	3120.00
	LP		956.69	1243.12
0.1	PP	727.20	792.00	1440.00
	LP	727.20	792.00	1141.18
1.0	PP	720.72	727.20	792.00
	LP	720.72	727.20	792.00
10.0	PP	720.77	727.72	727.20
	LP		727.72	727.20
∞	PP	720.00	720.00	720.00
	LP	720.00	720.00	720.00

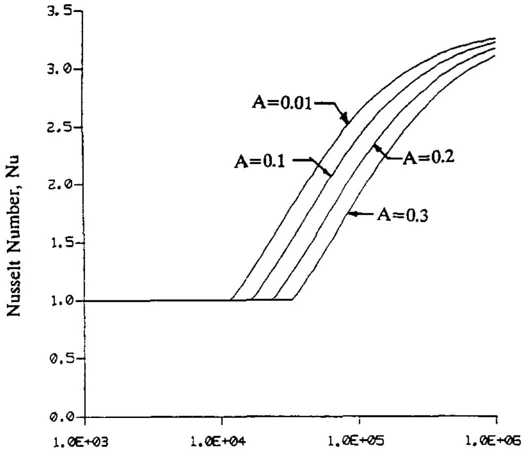

FIG. 4. Effect of mid-layer thickness on heat transfer for a two-layer system: $X_B = 1$.

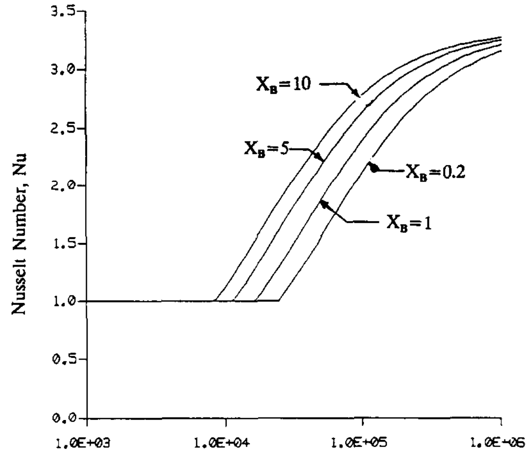

FIG. 5. Effect of conductivity on heat transfer for a two-layer system: $A = 0.1$.

As for heat transfer, the Nusselt number under the current case is

$$Nu = \frac{10Ra_1(1+AX_B)}{3Ra_1(1+AX_B)+5040\left(1+\frac{A}{X_B}\right)}. \tag{58}$$

Here Ra_1 is the Rayleigh number based on single fluid layer height h_1. The Nusselt number from equation (58) can also be written as a function of overall Rayleigh number, which is based on $h = (2h_1+h_2)$

$$Nu = \frac{10Ra(1+AX_B)}{3Ra(1+AX_B)+80\,640\left(1+\frac{A}{X_B}\right)(1+A)^4}. \tag{59}$$

At this point, there is no published data available to compare the above formula for the present uniform heat flux boundary condition. The Nusselt number of Lienhard and Catton [9] for uniformly heated and cooled boundary condition, however, can be used as a reference. Figure 4 shows the Nusselt number as a function of overall Rayleigh number at several values of A. Within the range of $10^4 < Ra < 10^5$, the Nusselt number falls in the range $1 < Nu < 3$, which is in the same range as that of ref. [9]. Figure 5 shows the effect of X_B on Nu at $A = 0.1$. Compared to Fig. 4 in ref. [9] the trend of Nu with X_B is just opposite, i.e. a higher X_B gives a higher Nusselt number in the present case. From equation (59) it is seen that when $A/X_B < 1$, Nusselt number is a strong function of AX_B. Indeed, in their 'engineering estimate' for heat transfer, Lienhard and Catton [9] obtained an expression for Nu, which was a function of AX_B only. This is consistent with the present prediction. Figures 4 and 5 show that increasing A is equivalent to decreasing X_B, or Nu depends on AX_B.

The increase of thermal stability by inserting a solid partition can be appreciated by comparing the critical heat flux for both cases. Without the solid partition, the critical heat flux is

$$q_c = 720\frac{k_1 v_1 \alpha_1}{g\beta_1 h^4}. \tag{60}$$

The ratio of q_c in equation (53) to that in equation (60) is

$$R = 16(1+A)^4\left(1+\frac{A}{X_B}\right). \tag{61}$$

We see that if the solid partition is thin enough ($h_2/h_1 \ll 1$), $A \to 0$, the critical heat flux is increased by 16 times. In the study by Catton and Lienhard [7] who considered the same problem but with an isothermal wall, and the one by Yang and Yang [10], who considered Bénard–Marangoni instability in a two-layer system, an increase in Rayleigh number of 16 times was found. This is consistent with the present study. As for the Nusselt number, it will be

$$Nu = \frac{10Ra}{3Ra+80\,640} \tag{62}$$

where Ra is based on h. At the same Rayleigh number beyond the critical heat flux, the heat transfer is reduced much more due to the insertion of a solid partition.

4.3. *Single fluid separated by multi-solid partitions*

By merely inserting a thin partition into the center of a fluid layer, the critical heat flux for the onset of motion can be increased by 16 times. What is interesting in this section is to find the efficiency of inserting $(M-1)$ *thin* solid partition into a fluid layer. Now the total number of fluid layers is M. Again, the critical heat flux will be

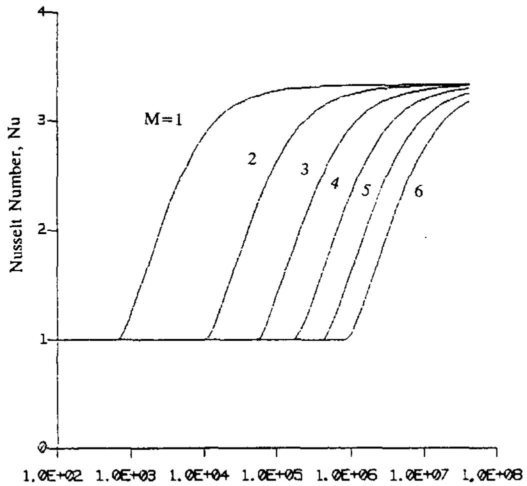

FIG. 6. Nusselt number dependence on Rayleigh number for a system with $(M-1)$ solid partitions.

$$q_c = 720\frac{k_1 \sum_{i=1}^{M} h_i}{\frac{g\beta_1}{\nu_1\alpha_1}\sum_{i=1}^{M} h_i^5}. \tag{63}$$

With the condition of

$$\sum_{i=1}^{M} h_i = h = \text{constant} \tag{64}$$

and

$$\frac{\partial q_c}{\partial h_i} = 0 \quad i = 1, 2, \ldots, M-1 \tag{65}$$

one can find that q_c reaches a maximum value only if

$$h_1 = h_2 = \cdots = h_i = \cdots = \frac{h}{M}. \tag{66}$$

Now the critical heat flux as from equation (63) is

$$q_c = 720M^3\frac{k_1\nu_1\alpha_1}{g\beta_1 h^4}. \tag{67}$$

The corresponding heat flux rate with Rayleigh number is

$$Nu = \frac{10Ra}{3Ra + 5040M^4}. \tag{68}$$

A plot representing the dependence of Nusselt number on Rayleigh number (based on h) at various M (here $M-1$ is the partition number, and M the number of fluid layers) is shown in Fig. 6. Equations (67) and (68), and Fig. 6 indicate that the increase in the critical heat flux is proportional to M^4, and Nusselt number is substantially reduced when M is large.

4.4. *Equivalent Rayleigh number*

It is always interesting to define an equivalent Rayleigh number for a multiple fluid layer system. By equation (43), if we define an equivalent Ra as the weighted average of k_ih_i of each layer, we will have

$$Ra = \frac{\sum_{i=1}^{N} Ra_i\, k_ih_i}{\sum_{i=1}^{N} k_ih_i}. \tag{69}$$

For solid partitions, $Ra_i = 0$, so that the solids have the role of reducing the equivalent Rayleigh number or stabilizing the flow. Apparently, only if the equivalent Rayleigh number reaches 720, will the fluid start to move from the rest state.

5. CONCLUSIONS

A closed form solution has been derived for a system consisting of an arbitrary number of fluid layers separated by solid partitions. The system is subjected to a uniform heat flux from below and adiabatic boundary conditions at two ends. The solution gives the critical heat flux, fluid velocities and heat transfer rate. When applying to a single and a double fluid layer, the critical Rayleigh numbers are consistent with the previously published solutions. The solid partitions are found to have optimum efficiency in suppressing fluid motion when they are uniformly distributed. The increase in critical heat flux is proportional to M^4 (where M is the number of fluid layers). An equivalent Rayleigh number can be defined as the weighted average of the product of thermal conductivity and height of each individual layer.

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INSTABILITE THERMIQUE ET TRANSFERT THERMIQUE DANS UN SYSTEME MULTICOUCHE SOUMIS A UN FLUX DE CHALEUR UNIFORME PAR LE BAS

Résumé—On étudie l'instabilité thermique et le transfert thermique dans un système consistant en plusieurs couches séparées par des partitions solides. Une solution analytique est obtenue quand le système est soumis à un flux uniforme par dessous et à des conditions aux limites adiabatiques aux deux extrémités. Pour les cas d'une et deux couches, les nombres de Rayleigh critiques s'accordent avec les résultats précédemment publiés. Les partitions solides sont trouvées être les plus efficaces pour supprimer l'écoulement et le transfert de chaleur quand elles sont également distribuées dans un seul fluide. Le nombre de Rayleigh équivalent est montré être la moyenne pondérée du produit de la conductivité thermique par la hauteur de chaque couche.

THERMISCHE INSTABILITÄT UND WÄRMEÜBERGANG IN EINEM MEHRSCHICHTSYSTEM BEI GLEICHFÖRMIGER BEHEIZUNG VON UNTEN

Zusammenfassung—Die thermische Instabilität und der Wärmeübergang in einem System aus mehreren durch feste Zwischenlagen getrennten Schichten wird untersucht. Es ergibt sich eine geschlossene Lösung für den Fall, daß das System durch einen gleichförmigen Wärmestrom von unten beheizt wird und an beiden Enden adiabate Randbedingungen herrschen. Für die Grenzfälle mit Einzel- und Doppelschichten stimmt die kritische Rayleigh-Zahl mit früheren Ergebnissen überein. Die festen Grenzschichten haben die beste Trenn- und Isolationswirkung für Strömung und Wärmeübergang bei gleichmäßiger Verteilung in einem einzelnen Fluid. Die äquivalente Rayleigh-Zahl erweist sich als der gewichtete Mittelwert des Produktes aus Wärmeleitfähigkeit und Höhe der einzelnen Schichten.

ТЕПЛОВАЯ НЕУСТОЙЧИВОСТЬ И ТЕПЛОПЕРЕНОС В МНОГОСЛОЙНОЙ СИСТЕМЕ, ПОДВЕРЖЕННОЙ ВОЗДЕЙСТВИЮ ОДНОРОДНОГО ТЕПЛОВОГО ПОТОКА СНИЗУ

Аннотация—Исследуется тепловая неустойчивость и теплоперенос в системе, которая состоит из множественных слоев, разделенных твердыми перегородками. Получено решение в замкнутой форме для случая, когда на систему воздействует снизу однородный тепловой поток и у двух торцов налагаются адиабатические граничные условия. В предельных случаях одного и двух слоев критические числа Рэлея хорошо согласуются с ранее опубликованными результатами. Найдено, что применение твердых перегородок наиболее эффективно для подавления течения и теплопереноса, когда они одинаково распределены в одной жидкости. Показано, что эквивалентное число Рэлея является взвешенным средним произведения коэффициента теплопроводности и высоты каждого слоя.